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SYNTHESIZING SUBOPTIMAL CONTROLS IN NON-LINEAR DYNAMICAL SYSTEMS†

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A non-linear dynamical system whose motion is described by second-order Lagrangian equations is considered. The following problem is investigated. It is required to construct a control in the form of a synthesis, i.e. in the form of a function of the current values of the phase coordinates and time, that takes the system in a given time from an arbitrary initial phase state to a given final phase state. A method for solving this problem is presented based on the use of first integrals of the equations of free motion of the system and local connection of the values of the required control forces in a small neighbourhood of the final moment of the control process. Here the control processes are synthesized analytically and are optimal in the sense of minimizing a functional of mixed type [1] over almost the entire time interval of the control process. The efficiency of the proposed method of control synthesis is illustrated by examples.

THE RESULTS obtained are a further development of investigations [2–5] concerned with the use of first integrals in optimal control problems and extend them to the case of control synthesis in variational problems where the ends of the phase trajectories are fixed.

1. Consider a multidimensional non-linear dynamical system whose motion is described by Lagrange equations of the second kind

$$\frac{d}{dt}\frac{\partial K}{\partial q_i^\cdot}-\frac{\partial K}{\partial q_i}=u_i+Q_i(\mathbf{q},\mathbf{q}^\cdot,t)\quad i=1,\ldots,n \tag{1.1}$$

Here $\mathbf{q}=(q_1,\ldots q_n)$ are generalized coordinates for the system, n is the number of its degrees of freedom, and the dot denotes differentiation with respect to time.

The generalized forces consist of control forces u_i that are to be determined together with the terms $Q_i(\mathbf{q},\mathbf{q}^\cdot,t)$ which consist of all the remaining outer and inner forces.

The kinetic energy of the system is given by the quadratic form

$$K(\mathbf{q},\mathbf{q}^\cdot)=\frac{1}{2}\sum_{i,j}A_{ij}(\mathbf{q})q_i^\cdot q_j^\cdot \tag{1.2}$$

where the A_{ij} are elements of a symmetric positive-definite $(n\times n)$-matrix $A(q)$ and the summation is carried out over values of i and j from 1 to n.

Using (1.1) and (1.2) we can reduce the equations of motion to the form

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$$A(\mathbf{q})\,\mathbf{q}^{\cdot\cdot} + B(\mathbf{q}, \mathbf{q}^{\cdot}, t) = \mathbf{u} \tag{1.3}$$

where $\mathbf{u} = (u_1, \ldots, u_n)$ is the vector of control forces and $B = (B_1, \ldots, B_n)$ is a known vector-valued function.

Introducing the auxiliary $2n$-dimensional vector $\mathbf{x} = (q_1, \ldots, q_n, q_1^{\cdot}, \ldots, q_n^{\cdot})$, the equations of motion (1.3) can be written in the form

$$\mathbf{x}^{\cdot} = \mathbf{f}(\mathbf{x}, t) + C(\mathbf{x})\,\mathbf{u} \tag{1.4}$$

Here $\mathbf{f}(\mathbf{x}, t)$ and $C(\mathbf{x})$ are known matrices of dimensions $(2n \times 1)$ and $(2n \times n)$, respectively.

Suppose that the initial time $(t = 0)$ and final time $(t = T)$ of the control process are specified, together with the boundary conditions

$$\mathbf{x}(0) = \mathbf{x}_0 \tag{1.5}$$

$$\mathbf{x}(T) = \mathbf{x}_T \tag{1.6}$$

We formulate the following control problem.

Problem C. Find a control in the form of a synthesis $\mathbf{u} = \mathbf{u}(\mathbf{x}, t)$ taking system (1.4) from an arbitrary initial phase state (1.5) to a given state (1.6) in a specified time $T < \infty$.

2. The idea behind the method presented below for solving problem C consists of the following. We shall assume that over almost the entire time interval of the control process, i.e. for $t \in [0, T - 2e]$, where e is a small positive number, the system moves under the action of control forces $\mathbf{u}^0(\mathbf{x}, t)$ which extremize some functional $J[\mathbf{x}, \mathbf{u}]$ for given differential constraints (1.4) and initial state (1.5). In a small neighbourhood of the end of the control process, i.e. for $t \in [T - 2e, T]$, the system is acted on in a suitable manner by connecting piecewise-constant control forces which ensure that the system arrives at the specified final state (1.6).

In accordance with the above we shall look for a solution of problem C in the form

$$\mathbf{u}(\mathbf{x}, t) = \begin{cases} \mathbf{u}^0(\mathbf{x}, t), & 0 \leqslant t \leqslant T - 2e \\ \mathbf{u}^-, & T - 2e < t < T - e \\ \mathbf{u}^+, & T - e < t \leqslant T \end{cases} \tag{2.1}$$

where $\mathbf{x}^0(\mathbf{x}, t)$ is an n-dimensional vector of control forces that is the solution of some variational problem with a free right-hand end of the phase trajectory for dynamical system (1.4), and $\mathbf{u}^- = (u_1^-, \ldots, u_n^-)$, $\mathbf{u}^+ = (u_1^+, \ldots, u_n^+)$ are n-dimensional constant vectors whose components are to be determined.

We will describe one of the possible procedures for choosing the control forces $\mathbf{u}^0(\mathbf{x}, t)$.

Suppose $v_1(\mathbf{x}, t), \ldots, v_m(\mathbf{x}, t)$ $(m \leqslant 2n)$ are first integrals of the equations of free motion of the dynamical system under consideration

$$\mathbf{x}^{\cdot} = \mathbf{f}(\mathbf{x}, t) \tag{2.2}$$

We choose an arbitrary differentiable function $W(y_1, \ldots, y_m)$ and consider a functional of the form

$$J[\mathbf{x}, \mathbf{u}, W] = W\{\mathbf{v}[\mathbf{x}(T_1), T_1]\} + \frac{1}{2}\int_0^{T_1} \sum_j \{k_j < \nabla_{\mathbf{x}} W[\mathbf{v}(\mathbf{x}, t)], C_j(\mathbf{x}) >\}^2 dt +$$
$$+ \frac{1}{2}\int_0^{T_1} \sum_j \left[\frac{u_j(\mathbf{x}, t)}{k_j}\right]^2 dt, \quad W[\mathbf{v}(\mathbf{x}, t)] = W[v_1(\mathbf{x}, t), \ldots, v_m(\mathbf{x}, t)] \tag{2.3}$$

Here k_j are given constants, $C_j(\mathbf{x})$ is the jth column-vector of the matrix $C(\mathbf{x})$, T_1 is the specified time of the control process, Δ_x is the gradient operator with respect to the variable x, and $\langle , \rangle$ represents the scalar product of vectors.

The first term of functional (2.3) (the terminal part) is a function of the phase coordinates at time $t=T_1$, the second describes properties of both the dynamical system (2.2) itself and its controlling apparatus. The third term of functional (2.3) can be interpreted as the cost of controlling the motion of the dynamical system [6, 7]. A more complete physical interpretation of the first two terms of the quality criterion (2.3) can be obtained for a specific choice of the function W and first integrals $v_i(\mathbf{x}, t)$. For example, in the case when system (2.2) is conservative and the function $W[\mathbf{v}(\mathbf{x}, t)]$ is the energy integral, the first term of function (2.3) defines the total mechanical energy of the system at time $t=T_1$, and the second describes the rate of dissipation of mechanical energy under the controlled motion of the system in question.

We will formulate an auxiliary control problem.

Problem A. Find a control in the form of a synthesis $\mathbf{u}^0=\mathbf{u}^0(\mathbf{x}, t)$ which minimizes functional (2.3) given the differential constraints (1.4) and initial state (1.5).

We know [2, 5] that the solution of problem A has the form

$$u_j^0(\mathbf{x}, t) = -k_j^2 \langle \nabla_x W[\mathbf{v}(\mathbf{x}, t)], C_j(\mathbf{x})\rangle, \ j=1, \ldots, n \tag{2.4}$$

Here the optimal motion of the system is given by the solution of the Cauchy problem (1.4), (1.5), (2.4), and we have the relation

$$\min_{\mathbf{u}} J[\mathbf{x}, \mathbf{u}, W] = J[\mathbf{x}, \mathbf{u}^0, W] = W[\mathbf{v}(\mathbf{x}_0, 0)] \tag{2.5}$$

We put $T_1=T-2e$ in formula (2.3) and denote by $\mathbf{x}(T-2e)$ the value at time $t=T-2e$ of the solution of the Cauchy problem (1.4), (1.5), (2.4). Suppose $q_i(T-2e)$ and $q_i^{\cdot}(T-2e)$ are the corresponding values of the generalized forces and their velocities, corresponding to values of the components $x_i(T-2e)$ and $x_{i+n}(T-2e)$.

We shall determine piecewise-constant controls u_i to take the dynamical system (1.3) over the time $2e$ from the state $q_i(T-2e)$, $q_i^{\cdot}(T-2e)$ to the state $q_1(T)$, $q_i^{\cdot}(T)$ $(i=1, \ldots, n)$ specified by boundary condition (1.6).

Using Taylor's formula for the functions $q_i(t)$ and $q_i^{\cdot}(T)$ we can write, up to terms of second order of smallness in e

$$\begin{aligned} q_i(T-e) &= q_i(T-2e) + eq_i^{\cdot}(T-2e) + \tfrac{1}{2} e^2 q_i^{\cdot\cdot}(T-2e) \\ q_i^{\cdot}(T-e) &= q_i^{\cdot}(T-2e) + eq_i^{\cdot\cdot}(T-2e), \ i=1, \ldots, n \end{aligned} \tag{2.6}$$

(where we mean by $q_i^{\cdot\cdot}(T-2e)$, in general, the appropriate one-sided second derivative).

On the other hand, given the same assumptions we also have the relations

$$\begin{aligned} q_i(T-e) &= q_i(T) - eq_i^{\cdot}(T) + \tfrac{1}{2} e^2 q_i^{\cdot\cdot}(T) \\ q_i(T-e) &= q_i(T) - eq_i^{\cdot\cdot}(T), \ i=1, \ldots, n \end{aligned} \tag{2.7}$$

We require continuity of the phase trajectory of the system at time $t=T-e$. This requirement reduces to the need to match the values of the controls in a neighbourhood of $t=T-e$ and enables one to determine the components u_i^-, u_i^+ $(i=1, \ldots, n)$. Equating the right-hand sides of (2.6) and (2.7) we have conditions of continuity for the phase trajectory, from which we determine the acceleration components

$$\begin{aligned} q_i^{\cdot\cdot}(T-2e) &= e^{-2}[q_i(T) - q_i(T-2e) - \tfrac{1}{2} eq_i^{\cdot}(T) - {}^3/_2\, eq_i^{\cdot}(T-2e)] \\ q_i^{\cdot\cdot}(T) &= e^{-2}[q_i(T-2e) - q_i(T) + \tfrac{1}{2} eq_i^{\cdot}(T-2e) + {}^3/_2\, eq_i^{\cdot}(T)], \ i=1, \ldots, n \end{aligned} \tag{2.8}$$

Note that the values of $q_i^{\cdot}(T-2e)$ calculated from formulae (2.8) are not, in general, equal to the corresponding accelerations of the system at time $t = T - 2e$ found by solving the Cauchy problem (1.4), (1.5), (2.4).

Having calculated $q_i^{\cdot\cdot}(T-2e)$ and $q_i^{\cdot\cdot}(T)$ from formulae (2.8), using the equations of motion (1.3) we obtain the final formulae for the required matching of the piecewise-continuous controls

$$u_i^- = \sum_{j=1}^{n} A_{ij}[\mathbf{q}(T-2e)]q_j^{\cdot\cdot}(T-2e) + B_i[\mathbf{q}(T-2e), \mathbf{q}^{\cdot}(T-2e), T-2e] \tag{2.9}$$

$$u_i^+ = \sum_{j=1}^{n} A_{ij}[\mathbf{q}(T)]q_j^{\cdot\cdot}(T) + B_i[\mathbf{q}(T), \mathbf{q}^{\cdot}(T), T], \quad i = 1, \ldots, n$$

On the basis of the above we obtain the following algorithm for solving problem C.

1. One or other method is used to determine first integrals $v_1(\mathbf{x}, t), \ldots, v_m(\mathbf{x}, t)$ of the equations of free motion (2.2).

2. A continuous differentiable function $W(y_1, \ldots, y_m)$ is chosen and the control $\mathbf{u}^0(\mathbf{x}, t)$ is calculated from formulae (2.4).

3. The Cauchy problem (1.4), (1.5), (2.4) is solved and the motion of the system in the time interval $t \in [0, T-2e]$ is calculated, where $0 < e << T$.

4. The accelerations $q_i^{\cdot\cdot}(T-2e)$ and $q_i^{\cdot\cdot}(T)$ $(i = 1, \ldots, n)$ are calculated using formulae (2.8) and, with the help of relations (2.9), the piecewise-constant controls $\mathbf{u}^-$ and $\mathbf{u}^+$ which ensure that the system arrives at the given final phase state (1.6) are calculated.

5. The motion of the system for $t \in [T-2e, T]$ is computed from the formulae

$$q_i(t) = q_i(T-2e) + (t - T + 2e)q_i^{\cdot}(T-2e) + \tfrac{1}{2}(t - T + 2e)^2 q_i^{\cdot\cdot}(T-2e) \tag{2.10}$$

for $T - 2e < t \leqslant T - e$ $(i = 1, \ldots, n)$ and from

$$q_i(t) = q_i(T) + (t - T + e)q_i^{\cdot}(T) + \tfrac{1}{2}(t - T + e)^2 q_i^{\cdot\cdot}(T) \tag{2.11}$$

for $T - e < t \leqslant T$ $(i = 1, \ldots, n)$.

6. The required control in the form of a synthesis solving problem C is determined from formulae (2.1), (2.4), (2.8) and (2.9).

The algorithm proposed provides the possibility of constructing a family of solutions of problem C depending both on the choice of first integrals v_i and on the choice of the function $W(y_1, \ldots, y_m)$. This arbitrariness can be used, for example, with the aim of further optimizing the solution of problem C with respect to some quality criterion. In particular, from the choice of the function $W(y_1, \ldots, y_m)$ it is clear that one can ensure that constraints on the required control are satisfied.

Analysis of formulae (2.8) and (2.9) shows that the controls u_i^- and u_i^+ are of order e^{-2} and so their contributions to functional (2.3) could be substantial. In this connection, it is desirable to choose the function $W(y_1, \ldots, y_m)$ so that $x(T-2e)$ belongs to a sufficiently small neighbourhood of the specified end state (1.6).

We remark that the case when $e = 0$ and (2.4) is used to calculate the control over the entire time interval $t \in [0, T]$ is also of interest. Then, naturally, there is no guarantee that with an arbitrarily chosen function $W(y_1, \ldots, y_m)$ the final condition (1.6) will be satisfied. However, one may hope that with a suitable choice of $W(y_1, \ldots, y_m)$ a control of form (2.4) could take the system under investigation to the given final state.

3. As an illustration of the efficiency of the algorithm proposed in Sec. 2 we consider a number of examples.

Example 1. Suppose that a point particle of mass m moves along a horizontal axis under the action of a force with potential $P(q)$ and a control $u(q, q^{\cdot}, t)$.

The equation of motion has the form

$$mq^{\cdot\cdot} = -dP(q)/dq + u(q, q^{\cdot}, t) \tag{3.1}$$

It is required to construct a control in the form of a synthesis $u(q, q^{\cdot}, t)$ taking the particle in a time $T < \infty$ from a given initial state

$$q(0) = q_0, \; q^{\cdot}(0) = q^{\cdot}_0 \tag{3.2}$$

to the final state

$$q(T) = q_T, \; q^{\cdot}(T) = q^{\cdot}_T \tag{3.3}$$

To solve this problem we apply the algorithm described in Sec. 2.

We introduce new variables $x_1 = q$ and $x_2 = q^{\cdot}$. In x_1, x_2 variables Eq. (3.1) and boundary conditions (3.2), (3.3) take the form

$$x^{\cdot}_1 = x_2, \; x^{\cdot}_2 = -(dP(x_1)/dx_1 + u(\mathrm{x}, t))/m \tag{3.4}$$

$$x_i(0) = x_{i0}, \; i = 1, 2 \tag{3.5}$$

$$x_i(T) = x_{iT}, \; i = 1, 2 \tag{3.6}$$

We shall look for a control $u(\mathbf{x}, t)$ of the form (2.1). With $u = 0$ Eqs (3.4) have a first integral—the energy integral

$$v(\mathrm{x}) = \tfrac{1}{2} m x_2^2 + P(x_1) \tag{3.7}$$

We put $T_1 = T - 2e$, $0 < e \ll T$, and $W[v(\mathbf{x})] = v(\mathbf{x})$, where $v(\mathbf{x})$ is given by (3.7). Then the function (2.3) becomes

$$J[\mathrm{x}, u, W] = \frac{1}{2} m x_2^2 (T - 2e) + P[x_1(T - 2e)] + \frac{1}{2} \int_0^{T-2e} x_2^2(t)\, dt + \frac{1}{2} \int_0^{T-2e} u^2(\mathrm{x}, t)\, dt \tag{3.8}$$

The end-point components of functional (3.8) give the value of the total mechanical energy of the system at time $t = T - 2e$, the third term in (3.8) describes the dissipation of mechanical energy under the controlled motion, and the fourth is the cost of the control.

We choose $u^{\cdot}(\mathbf{x}, t)$ in formula (2.1) so that the functional (3.8) is minimized given differential constraints (3.4) and initial state (3.5). This control is given by formula (2.4), and using (3.4) and (3.7) it can be written in the form

$$u^{\circ}(\mathrm{x}, t) = -x_2(t) \tag{3.9}$$

Suppose $q(T - 2e)$ and $q^{\cdot}(T - 2e)$ are the values of the generalized coordinate and its velocity corresponding to the value $\mathbf{x}(T - 2e)$ of the solution of the Cauchy problem (3.4), (3.5) with $u(\mathbf{x}, t) = u^{\cdot}(\mathbf{x}, t)$. Using formulae (2.6)–(2.8) for $i = 1$, we determine the acceleration components $q^{\cdot\cdot}(T - 2e)$ and $q^{\cdot\cdot}(T)$. With the help of formulae (2.9) for $i = 1$ and Eq. (3.1) for the matching values of the control u^-, u^+, we obtain the relations

$$u^- = mq^{\cdot\cdot}(T - 2e) + dP[q(T - 2e)]/dq, \; u^+ = mq^{\cdot\cdot}(T) + dP[q(T)]/dq \tag{3.10}$$

The motion of the system under the controls (2.1), (3.9) and (3.10) obtained for $t \in (T - 2e, T]$ is given by formulae (2.10) and (2.11) with $i = 1$.

Example 2. It is required to determine the control $u = u(\mathbf{x})$ taking a dynamical system of form

$$x^{\cdot}_1 = x_2, \; x^{\cdot}_2 = u \tag{3.11}$$

from a given initial phase state (3.5) to a given final state (3.6) in a given time $T < \infty$.

A solution of this problem can be obtained using the algorithm described in Sec. 2 for the case $e = 0$.

For $u = 0$ Eq. (3.1) has a first integral $v(\mathbf{x}) = x_2$. Consider a functional of the form

$$J[u, W] = W[x_2(T)] + \frac{1}{2} \int_0^T \left\{ \left(\frac{u}{k}\right)^2 + k^2 \left[\frac{\partial W(x_2)}{\partial x_2}\right]^2 \right\} dt \tag{3.12}$$

where $W(x_2)$ is a given continuously differentiable function and k is a given parameter.

We know [2, 5] that the control

$$u^\circ(x) = -k^2 \partial W(x_2)/\partial x_2 \tag{3.13}$$

provides an absolute minimum for the functional (3.12) under differential constraints (3.11) and the given initial state (3.5). The optimal motion of the system is given by the solution of the Cauchy problem (3.11), (3.5) with $u = u^\circ(x)$. We will show that the function $W(x_2)$ can be chosen so that this solution satisfies the final condition (3.6).

Suppose

$$W(x_2) = ax_2^2 + bx_2 \tag{3.14}$$

where a and b are parameters to be determined.

The solution of the Cauchy problem (3.11), (3.5), (3.13), (3.14) for $u = u^\circ(x)$ has the form

$$\begin{aligned} x_1(t) &= x_{10} - bt/(2a) + a_1(1 - \exp(-2k^2at)) \\ x_2(t) &= -b/(2a) + 2k^2aa_1\exp(-2k^2at);\ a_1 = (2ax_{20} + b)/(4k^2a^2) \end{aligned} \tag{3.15}$$

We require that conditions (3.6) be satisfied. Then we obtain from (3.15) and (3.6) relations linking the parameters a and b

$$\begin{aligned} &x_{10} + a_1(1 - \exp(-2k^2aT)) - bT/(2a) = x_{1T} \\ &2k^2aa_1\exp(-2k^2aT) = x_{2T} + b/(2a) \end{aligned} \tag{3.16}$$

For simplicity, we consider the case

$$x_{20} = 0,\ x_{2T} \neq 0 \tag{3.17}$$

Then the expressions

$$\begin{aligned} &b = 2ax_{2T}/(F(a) - 1),\ \exp(-2k^2aT) = F(a) \\ &F(a) = 1 - 2k^2x_{2T}T/(x_{2T}/a + 2k^2(x_{1T} - x_{10})) \end{aligned} \tag{3.18}$$

follow from (3.16) and (3.17).

Analysis of the behaviour of the functions $F(a)$ and $\exp(-2k^2aT)$ in the domain $a > 0$ shows that the second equation in (3.18) has a positive root $a = a^* > 0$ as long as the condition

$$Tx_{2T}/(x_{1T} - x_{10}) < 1 \tag{3.19}$$

is satisfied.

After determining a^* the parameter b is calculated from the first formula in (3.18). Thus a control in the form of the synthesis

$$u^\circ(x) = -k^2(2ax_2 + b)$$

where the parameters a and b are given by formulae (3.18), takes the dynamical system (3.11) under assumptions (3.17) and (3.19) from the initial state (3.5) to the final state (3.6) and provides an absolute minimum to the functional

$$J[u] = ax_2^2(T) + bx_2(T) + \frac{1}{2}\int_0^T [(\frac{u}{k})^2 + k^2(2ax_2 + b)^2]dt$$

Note that by formulae (2.5) and assumption (3.17) we have $J[u^0] = 0$.

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